Modeling highway-traffic headway distributions using superstatistics

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We study traffic clearance distributions (i.e., the instantaneous gap between successive vehicles) and time-headway distributions by applying the Beck and Cohen superstatistics. We model the transition from free phase to congested phase with the increase of vehicle density as a transition from the Poisson statistics to that of the random-matrix theory. We derive an analytic expression for the spacing distributions that interpolates from the Poisson distribution and Wigner's surmise and apply it to the distributions of the net distance and time gaps among the succeeding cars at different densities of traffic flow. The obtained distribution fits the experimental results for single-vehicle data of the Dutch freeway A9 and the German freeway A5.

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In empirical highway-traffic observations [1], it has been found that traffic can be either free or congested. In free-flow conditions, drivers can choose their own speed. The distances between cars are uncorrelated, and thus follow a Poisson distribution. Traffic congestion is a road condition characterized by slower speeds, longer trip times, and increased queueing. It occurs when traffic demand is greater than the capacity of a road (or of the intersections along the road). Kerner [2] classified the congestion regime into two distinct phases: synchronized flow and wide moving jams. In synchronized flow, the speeds of the vehicles are low and vary quite a lot between vehicles, but the traffic flow remains close to free flow. In wide moving jams, vehicle speeds are more equal and lower, and time delays can be quite large. Extreme traffic congestion, where vehicles are fully stopped for periods of time, is colloquially known as a traffic jam. Besides Kerner's three-phase theory, congested traffic has been described in terms of five congestion phases of different spatiotemporal properties, and their combinations [3,4]. A velocity dependent randomization variant of traffic cellular automata leads to the emergence of four distinct phases: free-flowing traffic, dilutely congested traffic, densely advancing traffic, and heavily congested traffic [5].

The question whether the transition from one regime to another is a smooth crossover or is a result of a genuine phase transition is still not settled in most traffic models. Measurements of traffic breakdown on various highways by Kerner and Rehborn [6] indicate a first-order transition between free-flow and synchronized traffic. Some traffic cellular automata models are suggested to exhibit phase transitions (see, e.g., [7] and references therein). However, the existence of such a transition can only be explicitly demonstrated in some limiting cases where certain dynamical processes are deterministic. Asymmetric chipping models, where a single particle hops to a nearest-neighbor site at a constant rate, suggest that the jamming phase transition does not take place. Rather, the system exhibits a smooth crossover between free-flow and jammed states, as the car density is increased [8].

Krbálek and Šeba studied the statistics of bus arrivals close to the Cuernavaca city center [9] where the distances between buses is optimized. They found that the Gaussian unitary ensemble (GUE) of random-matrix theory (RMT) [10]

$$P_{\mathrm{GUE}}(s) = \frac{32}{\pi^2}\frac{s^2}{D^3}e^{-4s^2/\pi D^2}, \tag{1}$$

where D is the mean spacing, successfully models the bus spacing distribution, and also the bus number variance measuring the fluctuations of the total number of buses arriving at a fixed location during a given time interval. RMT models the Hamiltonians of chaotic systems as members of an ensemble of random matices that depends only on the symmetry properties of the system. GUE models systems violating time reversal symmetry. Krbálek and Šeba justified the GUE properties of the bus arrival statistics by regarding the buses as one-dimensional interacting gas. Traffic is treated as a gas of interacting particles (vehicles) described by a distribution function with time evolution given by a Boltzmann equation. The steady state solution is given by the Boltzmann factor. The probability density function for the positions of the charges is given by $\exp(-\beta V)$ times a constant, where V and β are the potential energy and the inverse temperature of the gas, respectively. Dyson has shown several years ago that the exact level statistics of random-matrix ensembles is obtained for Coulomb interaction between the gas particles, when the car positions are identified with the eigenvalues of the ensemble matrices [10]. GUE corresponds to a value of $\beta=2$. Abul-Magd [11] found that the spacing distribution of GUE agrees with data measuring the gaps between parked cars on four streets in central London [12], where it is difficult to find a parking place. In light of these empirical findings, we expect GUE to reasonably describe traffic in the phase of wide (moving) jams, which are localized structures propagating upstream [2] with a mean spacing D between vehicles close to the minimum (safe) gap.

We shall therefore regard formula (1) as an empirical result for jammed traffic. On the other hand, the distances between cars if free-flow traffic are uncorrelated, and the spacing distribution, follow the Poisson distribution

$$P_{\mathrm{Poisson}}(s) = \frac{1}{D}e^{-s/D}. \tag{2}$$

Other forms of congested traffic that constitute synchronized flow [2], whose main characteristic is the apparent absence of a functional flow-density form, may be considered as a transition between (almost) regular dynamics and chaos. In this paper, we describe them using the concept of superstatistics (statistics of a statistics).

Superstatistics has been applied to model systems with partially chaotic classical dynamics within the framework of RMT in Refs. [13,14]. The formalism of superstatistics has been proposed by Beck and Cohen [15] as a possible generalization of statistical mechanics. The superstatistics concept is very general and has been applied to a variety of complex systems (see [16] and references therein). Essential for the superstatistical approach is the existence of an intensive parameter, which fluctuates on a large spatiotemporal scale compared to the fluctuations of the constituents of the system. In congested traffic, there is a relatively fast dynamics given by the velocity of the vehicle and a slow one given by the traffic density, which is spatiotemporally inhomogeneous. The two effects produce a superposition of two statistics, i.e., superstatistics. In fact, a traffic headway is inhomogeneous in space and in time. Effectively, it may consist of many spatial cells, where there are different values of the traffic density. Here the flow will always increase (decrease) with increasing density. In time series of flow-density measurements, the flow might increase or decrease, in sharp contrast to what is observed for the free-flow and jammed phase. The measured time series may consist of many time slices. This "irregularity" of the time series has been quantified by using the cross-correlation function [17] between density and flow. It is very close to 1 in free flow and the jams but almost 0 for synchronized flow. The order parameter introduced in [18] to reflect the degree of the internal interaction of vehicles takes a large value for synchronized flow, whereas free flow and the jam match its small values as weak mutual interactions.

Within the superstatistics framework, the car ensemble compromises various groups of cars jammed with different "local" mean spacing. We express any statistic P of a (sufficiently chaotic) congested traffic as an average of the corresponding statistic $P^{(G)}(D)$ for a Gaussian random ensemble over the local mean level spacing D. The superstatistical generalization is given by

$$P=\int_0^\infty f(D)P^{(G)}(D)dD. \tag{3}$$

Our goal in the present paper is to show that nearest-neighbor-spacing distribution (NNSD) obtained by substituting $P_{\mathrm{GUE}}(s)$ for $P^{(G)}(D)$ in Eq. (3), provides a plausible explanation for the observed car-spacing distribution at arbitrary traffic densities. The form of the probability distribution P as a weighted sum over equilibrium distributions $P^{(G)}$ up to now is based on purely phenomenological arguments. No fundamental derivation of Eq. (3) from the basic principles has been given. In this case the main problem is the relationship between the weights $f(D)$ of this expansion and specific random processes governing the system dynamics, which is currently the main direction of researches carried out in the field of superstatistics (see, e.g., Ref. [19]). Following Sattin [20], we evaluate $f(D)$ by using the principle of maximum entropy. Lacking a detailed information about the mechanism causing the deviation from the prediction of RMT, the most probable realization of $f(D)$ will be the one that extremizes the Shannon entropy $S=-\int_0^\infty f(D)\ln f(D)dD$ with the following constraints: (i) Basic for most statistical applications is the distinction between average quantities and their fluctuations. The fluctuation properties are defined in RMT for unfolded spectra, which have a unit mean level spacing. We thus require $\int_0^\infty f(D)DdD=1$. (ii) Vehicle headway distributions are a measure of narrow intervals of traffic densities. We require that mean level density of the superposed GUE's $\langle D^{-1}\rangle=\int_0^\infty f(D)D^{-1}dD$ is fixed. With these constraints, the maximization of S yields

$$f(D)=C(\alpha,D_0)\exp\left[-\alpha\left(\frac{D}{D_0}+\frac{D_0}{D}\right)\right], \tag{4}$$

where α and D_0 are parameters, which can be expressed in terms of the Lagrange multipliers of the constrained extremization, and $C(\alpha,D_0)=1/2D_0K_1(2\alpha)$, where $K_m(x)$ is a modified Bessel function of the second kind. The parameter D_0 is given by the condition of $\langle D\rangle=1$ as

$$D_0=\frac{K_1(2\alpha)}{K_2(2\alpha)}. \tag{5}$$

The parameter α defines the dispersion of the local mean spacing, whose variance is given by

$$\sigma^2=\frac{K_1(2\alpha)K_3(2\alpha)}{[K_2(2\alpha)]^2}-1 \approx \begin{cases} 1+[0.536\,274+4\ln(2\alpha)]\alpha^2+O(\alpha^3), & \text{for small } \alpha,\\ \dfrac{1}{2\alpha}+O\left(\dfrac{1}{\alpha^3}\right), & \text{for large } \alpha. \end{cases} \tag{6}$$

The distribution (4) tends to $\delta(D-1)$ at large α, which corresponds to traffic jams by assumption.

We note that the constraint (ii) imposed here is different from the corresponding constraint used in [14], which requires the existence of $\langle D^{-2}\rangle$. Namely the present choice has recently been found [21] to produce a distribution of local density of states $\nu=1/D$ which is very similar to the one obtained by Altshuler and Prigodin [22] for strictly one-dimensional disordered chains, which has been successfully applied in the numerical simulation of the closed wire (see, e.g., [23]). In addition, the present constraint (ii) is more suitable for analysis of traffic data, such as the data to be considered in this paper, where vehicle spacings are taken from separate intervals of traffic density.

We now apply superstatistics to calculate the NNSD for a system undergoing a transition out of chaos described by GUE statistics. For this purpose, we substitute Eqs. (5) and (6) into Eq. (4) to obtain

$$P(\alpha,s) = \frac{16}{\pi^2 D_0 K_1(2\alpha)} s^2 \times \int_0^\infty \exp\left[-\alpha\left(\frac{D}{D_0}+\frac{D_0}{D}\right) - \frac{4s^2}{\pi D^2}\right]\frac{dD}{D^3}, \quad (7)$$

where D_0 is given by Eq. (5).

We now try to model the transition of traffic from the free-flow regime to that of a moving jam as a dynamical transformation from the Poisson statistics to that of a GUE. Accordingly, the traffic headway NNSD at intermediate traffic densities has an intermediate behavior between the Poisson and GUE distributions (3) and (5). A quantitative interpretation of this model is provided by the superstatistical generalization of RMT. In the following, we show that the superstatistical spacing distribution in Eq. (7) is suitable for describing clearance distribution at arbitrary traffic density.

The distributions of space gap between vehicles (i.e., clearances in the traffic terminology) are recorded by double induction-loop detectors continuously during approximately 140 days on the Dutch two-lane freeway A9 [24]. The macroscopic traffic density was calculated for samples of $N=50$ subsequent cars passing a detector. The region of the measured densities $\rho \in [0,85 \text{ vehicle/km/lane}]$ is divided into 85 equidistant subintervals. The measured data in each density subinterval include, for each lane, the passage time of each vehicle, its velocity, and its length. From this, the individual bumper-to-bumper distance s_i among the succeeding cars [ith and $(i-1)$th] are determined. The bumper-to-bumper distance s_i among the succeeding cars [ith and $(i-1)$th] is calculated (after eliminating car-truck, truck-car, and truck-truck gaps). The mean distance among the cars is rescaled to 1 in all density regions. The car-spacing distributions of four density regions, which have been reported in Ref. [25] are shown in Fig. 1 by histograms. The curves are the best fit to the superstatistical distribution (7). The figure demonstrates the high quality of agreement between the proposed model and the experiment. The best-fit values of the superstatistical parameter α are 0.16, 0.48, 2.1, and 9.3, for density intervals centered around $\rho=0.5$, 4.5, 25.5, and 81.5 vehicle/km/lane, respectively. Interestingly, if we disregard the first interval where the spacing distribution is nearly Poissonian, we find that the best-fit values of α increases linearly with ρ such that

$$\alpha = c\rho, \quad c = 0.10 \pm 0.1. \quad (8)$$

In Ref. [25], the empirical clearance distributions are successfully compared with a one-dimensional thermodynamical particle gas model. The author considers a system of identical particles on the circumference of a circle. The particles interact with a repulsive potential inversely proportional to their mutual distance. The agreement between experimental and calculated distributions is obtained by varying one free parameter (inverse temperature β) that represents the traffic density. In spite of the success of this model, it is not obvious to us that (equilibrium) thermostatics can describe all of the different traffic phases in a similar way.

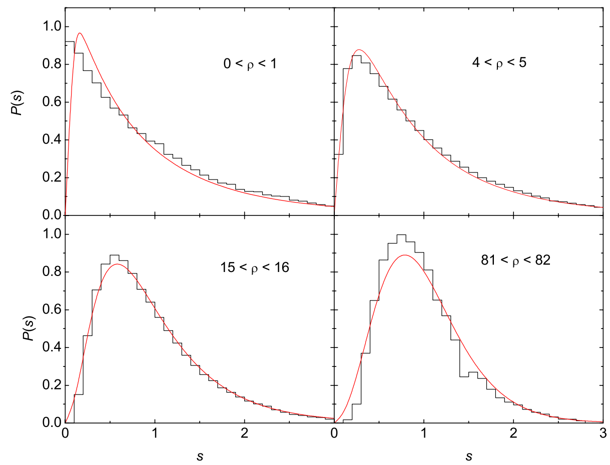

FIG. 1. (Color online) Probability density $P(s)$ for scaled spacing s between successive cars in traffic flow, taken from Ref. [25]. Histograms represent the clearance distributions computed for traffic data from a four density region, $\rho \in (0,1), (4,5), (25,26), (81,82)$ (in vehicle/km/lane). The curves represent the predictions of superstatistical model (8) where the best-fit values of the superstatistical parameter α are 0.16, 0.48, 2.1, and 9.3, respectively.

The second quantity we look at is the time-headway distribution, which is the time elapsing between two vehicles passing the detector. In principle, time headways are associated with space headways once the latter are measured in narrow vehicle-density intervals. The required single-vehicle data include, for each lane, the passage time t_i^0 of vehicle i, its velocity v_i, and its length l_i. From this, we determine the individual net to time gaps as $t_i = t_i^0 - t_{i-1}^0 - l_i/v_i$. Kerner *et al.* [26] measured single-vehicle data sets on the three-lane freeway section of the German freeway A5-South. They reported the time-headway distributions for the free-flow, synchronized flow, and moving jam phases. We calculated the mean values $\bar{t}$ of the net to time gaps in these distributions to be 1.52, 1.82, and 2.29 s, respectively. The distributions $P(\tau)$, where $\tau = t/\bar{t}$, are compared in Fig. 2 with the superstatistical distribution in Eq. (7). The agreement between the empirical and superstatistical distributions is very good. The best-fit values of the parameters α take large values (=5.9, 15, and 13, respectively) compared to those obtained in fitting the headway data for the same traffic phase. The corresponding values of the variance σ^2 of the parameter distribution $f(D)$ are quite small, being 0.084, 0.033, and 0.038, respectively. Due to the slow decrease of the variance σ^2 at large values of α [see last line of Eq. (6)], the distributions $P(\tau)$ look very similar for all three phases. A possible reason for the large values of α is the additional fluctuation of the mean time headway introduced by the mean-velocity fluctuation to that of the local mean space gap D. Surprisingly, the best-fit value of the superstatistical parameter α is almost the same for both synchronized flow and the jam ($\alpha=15$, 13, respectively), both have almost a GUE distribution.

To summarize, the space-gap distribution between vehicles in traffic jams shows strong "level repulsion" and is well reproduced by the Wigner surmise for GUE. The clear distances between vehicles in a free-flowing traffic, on the

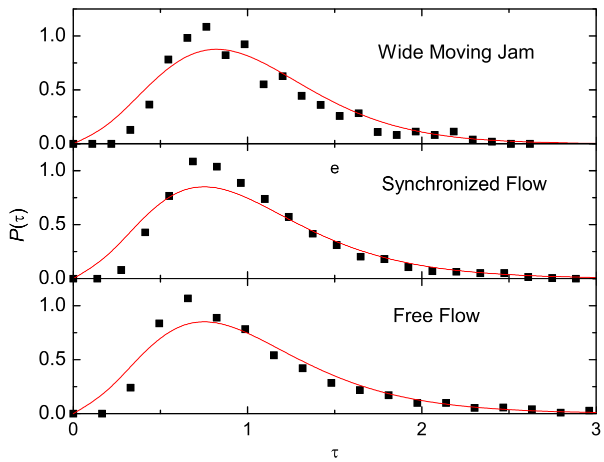

FIG. 2. (Color online) Probability density $P(\tau)$ for scaled net to time gaps τ between successive cars in traffic flow, taken from Ref. [26]. Histograms represent the time-headway distributions computed for traffic data for the free-flow, synchronized flow, and moving jam traffic phases. The curves represent the predictions of the superstatistical model (8) where the best-fit values of the superstatistical parameter α are 5.9, 15, and 13, respectively.

other hand, are uncorrelated and follow Poisson statistics. In this paper we use the concept of superstatistics to model congested traffic as a superposition of moving jams represented by GUE's with different mean level spacings. We derive an expression for NNSD that describes the transition between the Poisson-like statistics to that of a GUE by tuning a single parameter, namely the superstatistical parameter α. This parameter measures the variance of the fluctuating intensive variable (the mean distance between vehicles). We then apply the derived distribution to model transition of traffic from a stationary free-flow phase to a continuously growing congested nonstationary phase. Small values of α correspond to the free-flowing traffic, while the jammed and congested traffic phases are described by spacing distributions with large values of α. We found that the superstatistical NNSD provided a satisfactory description for the distance headway distributions at different densities by varying a single parameter. Thus, that the statistical features of traffic clearance exhibit a smooth crossover between a free-flow and a jammed state as the car density is increased. The best-fit values of superstatistical parameter increases almost linearly with the traffic density. This may suggest that single-vehicle data do not "feel" a first-order phase transition in traffic flow. However, it is not possible to draw a definite conclusion by considering so small number of cases. We note that the values of the inverse temperature of the one-dimensional gas model that fitted these and many other similar data [25] also do not show any stepwise variations as the car density increases. The superstatistical distribution was also successfully applied to time-headway distributions measured by Kerner and collaborators for the free-flow, synchronized flow, and moving jam traffic phases. While the mean time gaps between vehicles were essentially different in the three traffic phases, the fluctuation properties (measured by NNSD of "unfolded" levels) were almost the same for the two congested phases but rather different from the case of free flow.

The presented results support the possibility for applying the superstatistical RMT to the traffic systems. Obviously, the proposed model is no substitute for the elaborate investigations of the traffic problems. Nevertheless, the powerful methods of RMT may be useful in understanding some of its aspects even during the transition between the free-flow and congested phases.

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